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Dedicated to the memory of Eli Sternberg

SUMMARY

The notion of the driving traction on a surface of strain discontinuity in a continuum undergoing a general thermo-mechanical process is defined and discussed. In addition, the associated constitutive notion of a kinetic relation, in which the normal velocity of propagation of the surface of discontinuity may be a given function of the driving traction and temperature, is introduced for the special case of a thermoelastic material.

1. Introduction

Various aspects of the theory of finite elastostatics for materials characterized by non-elliptic elastic potentials have been studied in a number of recent papers; see for example Abeyaratne and Knowles (1987a,b, 1988a,b), Ball (1977) , Ball and James (1987), Ericksen (1975), Fosdick and MacSithigh (1983), Gurtin (1983), James (1981,1986), Knowles and Sternberg (1978), Silling (1988). One feature of such materials is that, under suitable conditions, they can sustain equilibrium deformations in which the displacement gradient and stress tensors suffer jump discontinuities across certain surfaces in the body, while the displacement and traction remain continuous; such singular surfaces have been called equilibrium shocks. One area in which the associated theory finds application is that of the continuum-mechanical modeling of a solid in equilibrium with more than one "phase" present. In this setting, an equilibrium shock corresponds to the boundary between two distinct phases of the material (Ball and James (1987), James (1981,1986), Silling (1988)).

A quasi-static motion involving equilibrium shocks at each instant may be dissipative (Knowles (1979)): in any portion of the body that is traversed by a moving shock, the rate of work of the external forces differs from the rate of storage of strain energy by the rate of work done in moving the surface of discontinuity. This latter rate of work can be expressed as the integral over the shock surface of the product of a scalar "driving traction" f with the component of shock velocity normal to the shock itself. At each instant during the motion, the value of f at each point on the singular surface can be calculated from the limiting values of the

deformation gradient on the two sides of the surface. If the driving traction f vanishes at all points of the shock for each equilibrium state visited during a quasi-static motion, the motion is dissipation-free. An equilibrium state for which f = 0 is said to satisfy the Maxwell condition.

In the presence of equilibrium shocks, the mechanical balance laws lead to the usual field equations at points in the body away from the singular surfaces and require the traction to be continuous across these surfaces. However, on solving even the simplest of boundary-value problems for non-elliptic elastic materials, one encounters a massive failure of uniqueness of solution (Abeyaratne (1980), Abeyaratne and Knowles (1987a)). One way to single out a preferred equilibrium field from among the infinitely many available ones is to require that the field be stable in the sense that the associated potential energy be an absolute minimum. If an equilibrium field containing a shock is to be stable in this sense, it is known that the Maxwell condition f = 0 is necessary (Abeyaratne (1983), Ericksen (1975), Gurtin (1983), James (1981)).

A somewhat more general point of view holds that the lack of uniqueness in the conventional equilibrium problems for non-elliptic elastic materials arises from a constitutive deficiency associated with particles on the shock surface. Such a view was adopted by Abeyaratne and Knowles (1988a,b) in the one-dimensional context of bar theory. By exploiting an analogy between the problem considered and internal-variable theories of inelastic solids (see, for example, Rice (1971,1975)), they are led to postulate a supplementary constitutive requirement in the form of a "kinetic relation"

between f and the velocity of the shock during a quasi-static motion. This leads to a determinate macroscopic response (or force-elongation relation) for the bar in quasi-static motions. In general, this response exhibits rate- and history-dependence. Two limiting cases of the kinetic relation describe rate-independent behavior: one corresponds to the Maxwell condition and hence to dissipation-free, reversible macroscopic response. The other leads to a force-elongation relation similar to that associated with rate-independent plasticity.

The investigations described above are for the most part carried out within the framework of the purely mechanical theory of non-elliptic elastic materials. They are limited to the study of equilibrium states or one-parameter families of such states (i.e., quasi-static motions), and they are often confined to one-dimensional settings as well. The purpose of the present study is to consider the corresponding issues in a more general, three-dimensional setting in which the material need not be elastic, and both thermal and inertial effects are taken into account. In Section 2, we recall the basic thermodynamic and mechanical laws and the associated field equations, inequalities and jump conditions for a continuum undergoing a thermo-mechanical process in which displacement and temperature are assumed to be continuous, but their gradients are permitted to jump across a moving surface. We derive a useful representation of the entropy production rate in Section 3, and we use it to introduce the notion of a driving traction acting on a singular surface in an arbitrary continuum during a thermo-mechanical process of the assumed type. We specialize the earlier results to isothermal dynamic processes in Section 4. No constitutive

assumptions are invoked until Section 5, where we specialize the foregoing results to the case of a continuum composed of a thermo-elastic material. Section 6 is devoted to a discussion of the notion of a constitutive "kinetic relation".

2. Balance laws, field equations and jump conditions

Consider a body B that occupies a region R in a reference configuration. A motion of the body on a time interval $[t_0,t_1]$ is characterized by a one-parameter family of invertible mappings $\hat{\underset{\sim}{y}}(\cdot,t)\colon R \to R_*$, with

$$\underset{\sim}{y} = \hat{\underset{\sim}{y}}(\underset{\sim}{x},t) = \underset{\sim}{x} + \underset{\sim}{u}(\underset{\sim}{x},t) \qquad \text{for } \underset{\sim}{x}\in R,\ t\in[t_0,t_1]. \tag{2.1}$$

We assume that the deformation $\hat{\underset{\sim}{y}}$, or equivalently the displacement $\underset{\sim}{u}$, is continuous with piecewise continuous first and second derivatives on $R\times[t_0,t_1]$. Let $\underset{\sim}{F}(\underset{\sim}{x},t) = \mathrm{Grad}\ \hat{\underset{\sim}{y}}(\underset{\sim}{x},t)$ and $\underset{\sim}{v}(\underset{\sim}{x},t) = \partial\hat{\underset{\sim}{y}}(\underset{\sim}{x},t)/\partial t$ stand respectively for the deformation gradient tensor and the particle velocity at points $(\underset{\sim}{x},t)$ in space-time where they exist.

Let $\rho(\underset{\sim}{x})$ denote the mass density of B at the point $\underset{\sim}{x}$ in the reference configuration, $\underset{\sim}{b}(\underset{\sim}{x},t)$ the body force per unit mass, and $\underset{\sim}{\sigma}(\underset{\sim}{x},t)$ the nominal stress tensor. At each t, we require $\rho(\cdot)$ and $\underset{\sim}{b}(\cdot,t)$ to be continuous on R, while $\underset{\sim}{\sigma}(\cdot,t)$ is to be piecewise continuous with a piecewise continuous gradient on R. The balance laws for linear and angular momentum require

$$\int_{\partial D} \underset{\sim}{\sigma}\underset{\sim}{n}\, dA + \int_D \rho\underset{\sim}{b}\, dV = d/dt \int_D \rho\underset{\sim}{v}\, dV\ , \tag{2.2}$$

$$\int_{\partial D} \hat{\underset{\sim}{y}} \times \underset{\sim}{\underset{\sim}{\sigma}}\underset{\sim}{n}\, dA + \int_{D} \hat{\underset{\sim}{y}} \times \rho \underset{\sim}{b}\, dV = d/dt \int_{D} \hat{\underset{\sim}{y}} \times \rho \underset{\sim}{v}\, dV , \tag{2.3}$$

respectively, at each $t \in [t_0, t_1]$ and for all regular subregions $D \subset R$.

Next, let $\underset{\sim}{q}(\underset{\sim}{x},t)$ denote the nominal heat flux vector, $r(\underset{\sim}{x},t)$ the heat supply per unit mass and $\varepsilon(\underset{\sim}{x},t)$ the internal energy per unit mass. At each t, we suppose that $r(\cdot,t)$ is continuous on R and that $\underset{\sim}{q}(\cdot,t)$ is piecewise continuous with a piecewise continuous gradient on R. The internal energy $\varepsilon(\cdot,\cdot)$ is required to be piecewise continuous with piecewise continuous first derivatives on $R \times [t_0, t_1]$. The first law of thermodynamics requires that at each instant t,

$$\int_{\partial D} \underset{\sim}{\underset{\sim}{\sigma}}\underset{\sim}{n} \cdot \underset{\sim}{v}\, dA + \int_{D} \rho \underset{\sim}{b} \cdot \underset{\sim}{v}\, dV + \int_{\partial D} \underset{\sim}{q} \cdot \underset{\sim}{n}\, dA + \int_{D} \rho r\, dV$$

$$= d/dt \int_{D} \rho \varepsilon\, dV + d/dt \int_{D} (1/2)\, \rho \underset{\sim}{v} \cdot \underset{\sim}{v}\, dV, \tag{2.4}$$

for every regular $D \subset R$. Finally, let $\theta(\underset{\sim}{x},t)$ denote the absolute temperature and $\eta(\underset{\sim}{x},t)$ the entropy per unit mass. At each t, we assume that $\theta(\cdot,t)$ is continuous with a piecewise continuous gradient on R, while $\eta(\cdot,\cdot)$ is assumed to be piecewise continuous with piecewise continuous first derivatives on $R \times [t_0, t_1]$. The rate of entropy production in a regular region $D \subset R$

is defined to be

$$\Gamma(t;D) = d/dt \int_D \rho\eta \, dV - \int_{\partial D} \underset{\sim}{q}\cdot\underset{\sim}{n}/\theta \, dA - \int_D \rho r/\theta \, dV. \tag{2.5}$$

The Clausius-Duhem version of the second law of thermodynamics requires

$$\Gamma(t;D) \geq 0 \quad \text{for } D \subset R, \; t\in[t_0,t_1]. \tag{2.6}$$

At a fixed instant t, localization of the balance laws (2.2)-(2.4) and the inequality (2.6) at a point $\underset{\sim}{x}$ at which $\underset{\sim}{F}$, $\dot{\underset{\sim}{F}}$, $\underset{\sim}{v}$, $\underset{\sim}{q}$, $\underset{\sim}{\sigma}$, ε and η are all continuous yields the following familiar local results:

$$\left.\begin{aligned} \text{Div } \underset{\sim}{\sigma} + \rho\underset{\sim}{b} &= \rho\dot{\underset{\sim}{v}}, \\ \underset{\sim}{\sigma}\underset{\sim}{F}^T &= \underset{\sim}{F}\underset{\sim}{\sigma}^T, \\ \underset{\sim}{\sigma}\cdot\dot{\underset{\sim}{F}} + \text{Div } \underset{\sim}{q} + \rho r &= \rho\dot{\varepsilon}, \\ \text{Div}(\underset{\sim}{q}/\theta) + \rho r/\theta &\leq \rho\dot{\eta}. \end{aligned}\right\} \tag{2.7}$$

On the other hand, suppose that S(t) is a regular surface in R at time t across which some or all of the thermo-mechanical quantities listed above suffer jump discontinuities. Localization of (2.2)-(2.6) at a point $\underset{\sim}{x}$ on S(t) yields the following jump conditions:

$$\left.\begin{aligned} &[[\underset{\sim}{\sigma}\underset{\sim}{n}]] + \rho\,[[\underset{\sim}{v}]]\,V_n = \underset{\sim}{0}, \\ &[[\underset{\sim}{\sigma}\underset{\sim}{n}\cdot\underset{\sim}{v}]] + [[\rho(\varepsilon + \underset{\sim}{v}\cdot\underset{\sim}{v}/2)]]\,V_n + [[\underset{\sim}{q}\cdot\underset{\sim}{n}]] = 0, \\ &[[\rho\eta]]\,V_n + [[\underset{\sim}{q}\cdot\underset{\sim}{n}/\theta]] \leq 0, \end{aligned}\right\} \text{ on } S(t), \tag{2.8}$$

where

$$V_n = \underset{\sim}{V}\cdot\underset{\sim}{n}, \tag{2.9}$$

and $\underset{\sim}{V} = \underset{\sim}{V}(\underset{\sim}{x},t)$ is the velocity of the point $\underset{\sim}{x}$ on the moving surface S(t). The unit normal $\underset{\sim}{n}$ on the singular surface S(t) is chosen such that $V_n \geq 0$; if $V_n > 0$, the positive side of S(t) is the side into which $\underset{\sim}{V}$ (and therefore $\underset{\sim}{n}$) points. If $g(\underset{\sim}{x},t)$ denotes a generic field quantity that jumps across S(t), we write $[[g(\underset{\sim}{x},t)]] = \overset{+}{g}(\underset{\sim}{x},t) - \overset{-}{g}(\underset{\sim}{x},t)$, where $\overset{+}{g}(\underset{\sim}{x},t)$ and $\overset{-}{g}(\underset{\sim}{x},t)$ stand for the limiting values of g at the point $\underset{\sim}{x}$ on S(t) from the positive and negative sides, respectively.

In addition to the jump conditions listed in (2.8), one also has the kinematic results

$$[[\underset{\sim}{F}]]\,\underset{\sim}{\ell} = \underset{\sim}{0}\,, \quad [[\underset{\sim}{v}]] = -\,[[\underset{\sim}{F}]]\,\underset{\sim}{V} \quad \text{on S(t)}\,, \tag{2.10}$$

where $\underset{\sim}{\ell}$ is any vector tangent to the singular surface S(t). The jump conditions in (2.10) are immediate consequences of the smoothness requirements imposed on the deformation (2.1).

Conversely, the field equations (2.7) together with the jump conditions (2.8), (2.10) imply the global balance laws (2.2), (2.3), (2.4), (2.6). All of the above results may be found in Truesdell and Noll (1965).

3. Shock driving tracton.

In the present section, we first introduce the notion of the shock driving traction associated with a thermo-mechanical process in an arbitrary continuum. The process is assumed to possess the smoothness specified in the preceding section; in particular, both displacement and temperature are required to be continuous across the moving singular surface S(t). We then make use of the shock driving traction to derive an alternate form for the entropy and energy jump conditions $(2.8)_3$ and $(2.8)_2$, respectively.

The concept of shock driving traction emerges naturally from an alternate representation for the rate of entropy production defined in (2.5). To obtain this representation, we begin by putting the energy jump condition $(2.8)_2$ into a form more useful for our purposes. In the identity

$$[[\underset{\sim}{\underset{\sim}{\sigma}}\underset{\sim}{n}\cdot\underset{\sim}{v}]] = (1/2)(\overset{+}{\underset{\sim}{v}} + \overset{-}{\underset{\sim}{v}})\cdot[[\underset{\sim}{\underset{\sim}{\sigma}}\underset{\sim}{n}]] + (1/2)(\overset{+}{\underset{\sim}{\underset{\sim}{\sigma}}}\underset{\sim}{n} + \overset{-}{\underset{\sim}{\underset{\sim}{\sigma}}}\underset{\sim}{n})\cdot[[\underset{\sim}{v}]] \ , \tag{3.1}$$

we replace $[[\underset{\sim}{\underset{\sim}{\sigma}}\underset{\sim}{n}]]$ and $[[\underset{\sim}{v}]]$ on the right by substituting from $(2.8)_1$ and $(2.10)_2$, respectively. This gives

$$[[\underset{\sim}{\underset{\sim}{\sigma}}\underset{\sim}{n}\cdot\underset{\sim}{v}]] = -(1/2)\ \rho\ [[\underset{\sim}{v}\cdot\underset{\sim}{v}]]\ V_n - (1/2)\ (\overset{+}{\underset{\sim}{\underset{\sim}{\sigma}}}\underset{\sim}{n} + \overset{-}{\underset{\sim}{\underset{\sim}{\sigma}}}\underset{\sim}{n})\cdot([[\underset{\sim}{F}]]\ \underset{\sim}{V}) \ . \tag{3.2}$$

By making use of $(2.10)_1$, one can show that

$$(\overset{+}{\underset{\sim}{\underset{\sim}{\sigma}}}\underset{\sim}{n} + \overset{-}{\underset{\sim}{\underset{\sim}{\sigma}}}\underset{\sim}{n})\cdot([[\underset{\sim}{F}]]\ \underset{\sim}{V}) = V_n\ (\overset{+}{\underset{\sim}{q}} + \overset{-}{\underset{\sim}{q}})\cdot[[\underset{\sim}{F}]] \ , \tag{3.3}$$

where the dot product on the right is that associated with a pair of tensors: $\underset{\sim}{A}\cdot\underset{\sim}{B} = \text{Trace}\,(\underset{\sim}{A}\underset{\sim}{B}^{T})$. Combining (3.2) and (3.3) yields

$$[[\varrho\underset{\sim}{n}\cdot\underset{\sim}{v}]] = -\,(1/2)\,\rho\,V_n\,[[\underset{\sim}{v}\cdot\underset{\sim}{v}]] - (1/2)\,V_n\,(\overset{+}{\underset{\sim}{g}} + \overset{-}{\underset{\sim}{g}})\cdot[[\underset{\sim}{F}]] \quad \text{on } S(t). \tag{3.4}$$

Using (3.4) to replace $[[\varrho\underset{\sim}{n}\cdot\underset{\sim}{v}]]$ in $(2.8)_2$ then supplies the alternate version of the energy jump condition:

$$\rho\,V_n\,[[\varepsilon]] = (1/2)\,V_n\,(\overset{+}{\underset{\sim}{g}} + \overset{-}{\underset{\sim}{g}})\cdot[[\underset{\sim}{F}]] - [[\underset{\sim}{q}\cdot\underset{\sim}{n}]] \quad \text{on } S(t). \tag{3.5}$$

We turn now to (2.5) and assume that, at each instant t, the region $D\subset R$ is divided into two parts $D^{+}(t)$ and $D^{-}(t)$ by the moving surface of discontinuity S(t). Allowing for the jump in specific entropy across S(t), we may write (2.5) as

$$\Gamma(t;D) = \int_D \rho\dot{\eta}\,dV - \int_{S(t)\cap D} [[\rho\eta]]\,V_n\,dA - \int_{\partial D} \underset{\sim}{q}\cdot\underset{\sim}{n}/\theta\,dA - \int_D \rho r/\theta\,dV\,, \tag{3.6}$$

which is equivalent to

$$\Gamma(t;D) = \int_D \rho\dot{\eta}\,dV - \int_{S(t)\cap D} [[\rho\eta]]\,V_n\,dA - \int_{\overset{+}{\partial D}\cup\overset{-}{\partial D}} \underset{\sim}{q}\cdot\underset{\sim}{n}/\theta\,dA - \int_{S(t)\cap D} [[\underset{\sim}{q}\cdot\underset{\sim}{n}/\theta]]\,dA$$
$$- \int_D \rho r/\theta\,dV\,. \tag{3.7}$$

Applying the divergence theorem to the third integral on the right in (3.7) and then utilizing $(2.7)_3$ to eliminate Div q yields

$$\Gamma(t;D) = \int_D \left\{ \frac{\rho\dot{\eta}\theta + \underset{\sim}{\sigma} \cdot \dot{\underset{\sim}{F}} - \rho\dot{\varepsilon}}{\theta} + \frac{\underset{\sim}{q} \cdot \text{Grad } \theta}{\theta^2} \right\} dV$$

$$- \int_{S(t)\cap D} \left\{ \frac{[[\underset{\sim}{q} \cdot \underset{\sim}{n}]] + [[\rho\eta\theta]] V_n}{\theta} \right\} dA , \qquad (3.8)$$

where we have made use of the continuity of the temperature θ across S(t). Finally, version (3.5) of the energy jump condition may be used to eliminate the term $[[\underset{\sim}{q}\cdot\underset{\sim}{n}]]$ in (3.8), yielding the desired representation for the rate of entropy production:

$$\Gamma(t;D) = \Gamma_{loc}(t;D) + \Gamma_{con}(t;D) + \Gamma_s(t;D) , \qquad (3.9)$$

where

$$\Gamma_{loc}(t;D) = \int_D \left\{ \frac{\rho\dot{\eta}\theta + \underset{\sim}{\sigma} \cdot \dot{\underset{\sim}{F}} - \rho\dot{\varepsilon}}{\theta} \right\} dV, \qquad (3.10)$$

$$\Gamma_{con}(t;D) = \int_D \frac{\underset{\sim}{q} \cdot \text{Grad } \theta}{\theta^2} \, dV , \qquad (3.11)$$

$$\Gamma_s(t;D) = \int_{S(t)\cap D} \left\{ \frac{[[\ \rho\varepsilon - \rho\eta\theta - (\overset{+}{\underset{\sim}{q}} + \overset{-}{\underset{\sim}{q}}) \cdot \underset{\sim}{F}/2\ ]]\ V_n}{\theta} \right\} dA \ . \tag{3.12}$$

In (3.9), the total rate of entropy production $\Gamma(t;D)$ at the instant t for the subregion $D \subset R$ is decomposed into three parts: Γ_{loc} arises from local dissipation in the material away from the singular surface; Γ_{con} is the entropy production rate due to heat conduction; finally, Γ_s represents the contribution to the entropy production rate arising from the moving singular surface S(t). A similar decomposition in the absence of a surface of discontinuity is given by Truesdell and Noll (1965), § 79.

The <u>internal dissipation</u> $\delta(\underset{\sim}{x},t)$ is given by

$$\delta = \theta\dot{\eta} + (1/\rho)\underset{\sim}{q} \cdot \dot{\underset{\sim}{F}} - \dot{\varepsilon} \ ; \tag{3.13}$$

see Chapter 2 of Truesdell (1969). The local entropy production rate $\Gamma_{loc}(t;D)$ of (3.10) may be written in terms of δ as follows:

$$\Gamma_{loc}(t;D) = \int_D \rho(\underset{\sim}{x})\delta(\underset{\sim}{x},t)/\theta(\underset{\sim}{x},t)\ dV \ . \tag{3.14}$$

Next, it is convenient to introduce the Helmholtz free energy per unit mass $\psi(\underset{\sim}{x},t)$ defined by

$$\psi(\underset{\sim}{x},t) = \varepsilon(\underset{\sim}{x},t) - \theta(\underset{\sim}{x},t)\eta(\underset{\sim}{x},t) \ , \quad \underset{\sim}{x}\in R, \ t\in[t_0,t_1] \ . \tag{3.15}$$

We now define the (scalar) shock driving traction $f(\underset{\sim}{x},t)$ on the singular surface $S(t)$ by

$$f(\underset{\sim}{x},t) = \rho(\underset{\sim}{x})[[\psi(\underset{\sim}{x},t)]] - (1/2)\left(\overset{+}{\underset{\sim}{g}}(\underset{\sim}{x},t)+\overset{-}{\underset{\sim}{g}}(\underset{\sim}{x},t)\right)\cdot[[\underset{\sim}{F}(\underset{\sim}{x},t)]],$$

$$\underset{\sim}{x}\in S(t),\ t\in[t_0,t_1]\ . \qquad (3.16)$$

By (3.15), (3.16) and (3.12), we may rewrite the contribution $\Gamma_s(t;D)$ to the rate of entropy production due to the moving singular surface in terms of the shock driving traction $f(\underset{\sim}{x},t)$, the temperature $\theta(\underset{\sim}{x},t)$ on $S(t)$ and the normal velocity $V_n(\underset{\sim}{x},t)$ of $S(t)$:

$$\Gamma_s(t;D) = \int_{S(t)\cap D} f(\underset{\sim}{x},t)\ V_n(\underset{\sim}{x},t)/\theta(\underset{\sim}{x},t)\ dA. \qquad (3.17)$$

We note from (3.16) that, if the thermo-mechanical process under consideration is smooth in the sense that the free energy ψ and the deformation gradient $\underset{\sim}{F}$ are continuous everywhere at all times, then the shock traction f on any surface vanishes.

Employing the representation for the entropy production rate furnished by (3.9), (3.11), (3.14) and (3.17) in the Clausius-Duhem inequality (2.6) and localizing the result at a point $\underset{\sim}{x}$ away from the singular surface $S(t)$ yields the inequality

$$\rho\delta + (1/\theta)\underset{\sim}{q}\cdot\text{Grad}\ \theta \geq 0 \quad \text{on} \quad R\text{-}S(t),\ t\in[t_0,t_1]. \qquad (3.18)$$

This result, which can also be derived directly from $(2.7)_3$, $(2.7)_4$ and (3.13) and which in fact may be used in place of $(2.7)_4$, may be found in Truesdell (1969), p.34, Eq. (2.47). If the localization of the Clausius-Duhem inequality is instead carried out at a point $\underset{\sim}{x}$ on the singular surface, the result is the condition

$$f\, V_n \geq 0 \quad \text{on } S(t),\ t\in[t_0,t_1]\ , \tag{3.19}$$

which may also be derived directly from $(2.8)_3$ with the help of (3.5), (3.15) and (3.16). For the special case of isothermal, quasi-static processes in thermoelastic materials, the counterpart of (3.19) was obtained by Knowles (1979).

Conversely, if the local results (3.18) and (3.19) hold, then it follows from (3.10)-(3.12) that

$$\Gamma_{loc}(t;D) + \Gamma_{con}(t;D) \geq 0\ , \quad \Gamma_s(t;D) \geq 0\ , \tag{3.20}$$

and hence from (3.9) that the Clausius - Duhem inequality holds. Also, it is clear that the entropy production rate Γ vanishes for every sub-region D if and only if equality holds in both (3.18) and (3.19). If the inequality in (3.19) is strict on $D\cap S(t)$, we may conclude from (3.9) and (3.17) that the shock traction associated with the moving singular surface makes a positive contribution to the rate of entropy production $\Gamma(t;D)$ and thus represents a dissipative effect.

The alternate version (3.5) of the original energy jump condition $(2.8)_2$ may be rewritten in still another form with the help of (3.15) and (3.16):

$$\rho\theta[[\eta]]\, V_n = -\, f\, V_n - [[\underset{\sim}{q}\cdot\underset{\sim}{n}]] \quad \text{on } S(t),\ t\in[t_0,t_1]. \tag{3.21}$$

The original set of jump conditions (2.8) may be replaced by $(2.8)_1$, (3.21) and (3.19).

Note that no constitutive assumptions have yet been made. The foregoing analysis, however, would <u>not</u> apply to the classical adiabatic theory of shock waves in gas dynamics (see, e.g., Courant and Friedrichs (1948)), since we assume that the temperature is continuous across the singular surface S(t).

4. <u>Isothermal processes in a continuum</u>.

In this section, we specialize the foregoing results to the case of isothermal processes, by which we mean those for which

$$\theta(\underset{\sim}{x},t) = \theta_0 = \text{constant} \quad \text{for } \underset{\sim}{x}\in R,\ t\in[t_0,t_1]\ . \tag{4.1}$$

When (4.1) holds, the global form (2.4) of the first law can be used to eliminate the heat flux and heat supply terms from the original representation (2.5) for the entropy production rate. After doing so and making use of the definition (3.15) of the free energy ψ, one finds that

$$\Gamma(t;D) = (1/\theta_0)\left\{\int_{\partial D} \underset{\sim}{\sigma}\underset{\sim}{n}\cdot\underset{\sim}{v}\ dA + \int_D \rho\underset{\sim}{b}\cdot\underset{\sim}{v}\ dV - d/dt\int_D \rho(\psi + \underset{\sim}{v}\cdot\underset{\sim}{v}/2)\ dV\right\}. \quad (4.2)$$

Since the contents of the braces in (4.2) represent the excess of the rate of mechanical work over the rate of increase of the sum of free and kinetic energies associated with D, one infers from (4.2) that, for isothermal processes, the entropy production rate coincides with the rate of mechanical dissipation per unit temperature.

On the other hand, with (4.1) in force, (3.11) shows that the contribution $\Gamma_{con}(t;D)$ to $\Gamma(t;D)$ vanishes, so that by (3.9), (3.14) and (3.17), we also have

$$\Gamma(t;D) = (1/\theta_0)\left\{\int_D \rho\delta\ dV + \int_{S(t)\cap D} f\ V_n\ dA\right\}. \quad (4.3)$$

Comparison of (4.2) and (4.3) yields the identity

$$\int_{\partial D} \underset{\sim}{\sigma}\underset{\sim}{n}\cdot\underset{\sim}{v}\ dA + \int_D \rho\underset{\sim}{b}\cdot\underset{\sim}{v}\ dV + \int_{S(t)\cap D} (-f)\ V_n\ dA =$$

$$\int_D \rho\delta\ dV + d/dt\int_D \rho(\psi + \underset{\sim}{v}\cdot\underset{\sim}{v}/2)\ dV \quad \text{for } D\subset R,\ t\in[t_0,t_1]. \quad (4.4)$$

This mechanical balance motivates our choice of the name shock driving traction for f: Equation (4.4) suggests that $-f(\underset{\sim}{x},t)$ be interpreted as the component of a vector traction acting normal to the singular surface S(t) and exerted by the surface on the body at the point $\underset{\sim}{x}$ at time t. Equivalently, $f(\underset{\sim}{x},t)$ may be regarded as a normal traction applied to S(t) by the body.

5. Thermoelastic materials.

5.1. General processes in thermoelastic materials. The preceding discussion has made no use of special constitutive relations for the continuum under consideration. We now assume that, for the material at hand, there is a characterizing internal energy potential $\hat{\varepsilon}(\underset{\sim}{F},\eta)$ such that

$$\left.\begin{aligned} \varepsilon(\underset{\sim}{x},t) &= \hat{\varepsilon}(\underset{\sim}{F}(\underset{\sim}{x},t),\eta(\underset{\sim}{x},t)), \\ \underset{\sim}{\sigma}(\underset{\sim}{x},t) &= \rho\hat{\varepsilon}_{\underset{\sim}{F}}(\underset{\sim}{F}(\underset{\sim}{x},t),\eta(\underset{\sim}{x},t)), \\ \theta(\underset{\sim}{x},t) &= \hat{\varepsilon}_{\eta}(\underset{\sim}{F}(\underset{\sim}{x},t),\eta(\underset{\sim}{x},t)); \end{aligned}\right\} \tag{5.1}$$

moreover, it is assumed that $\hat{\varepsilon}_{\eta}(\underset{\sim}{F},\cdot)$ is invertible for every tensor $\underset{\sim}{F}$ with positive determinant. We call such a material thermoelastic. To avoid cumbersome formulas, we have assumed that ρ and $\varepsilon(\cdot,\cdot)$ are independent of $\underset{\sim}{x}$, so that the body is homogeneous in the reference configuration.

For a thermoelastic material, (5.1) imply that

$$\rho\dot{\varepsilon} = \underset{\sim}{\sigma}\cdot\dot{\underset{\sim}{F}} + \rho\theta\dot{\eta} \quad \text{on R-S(t)} , \tag{5.2}$$

so that the internal dissipation δ of (3.13) vanishes. It then follows from (3.14) that

$$\Gamma_{loc}(t;D) = 0 \tag{5.3}$$

for all t and all DCR, whence (3.9) becomes

$$\Gamma(t;D) = \Gamma_{con}(t;D) + \Gamma_s(t;D), \tag{5.4}$$

where Γ_{con} and Γ_s are given by (3.11) and (3.17), respectively. Thus in <u>every</u> process in a thermoelastic material, entropy production is due only to heat conduction and the motion of the singular surface S(t).

By $(5.1)_3$ and the assumed invertibility of $\hat{\varepsilon}_\eta(\underset{\sim}{F},\cdot)$, we may write $\eta = \hat{\eta}(\underset{\sim}{F},\theta)$ and hence introduce the Helmholtz free energy potential $\hat{\psi}$ for a thermoelastic material through

$$\hat{\psi}(\underset{\sim}{F},\theta) = \hat{\varepsilon}(\underset{\sim}{F},\hat{\eta}(\underset{\sim}{F},\theta)) - \hat{\eta}(\underset{\sim}{F},\theta)\theta \ . \tag{5.5}$$

Then by (3.15), $\psi(\underset{\sim}{x},t) = \hat{\psi}(\underset{\sim}{F}(\underset{\sim}{x},t),\theta(\underset{\sim}{x},t))$, and (5.1), (5.5) show that, for any process,

$$\underset{\sim}{\sigma}(\underset{\sim}{x},t) = \rho\, \hat{\psi}_{\underset{\sim}{F}}\,(\underset{\sim}{F}(\underset{\sim}{x},t),\theta(\underset{\sim}{x},t)), \quad \eta(\underset{\sim}{x},t) = -\hat{\psi}_\theta(\underset{\sim}{F}(\underset{\sim}{x},t),\theta(\underset{\sim}{x},t)). \tag{5.6}$$

For a thermoelastic material, the shock driving traction $\underset{\sim}{f}(\underset{\sim}{x},t)$ intro-

duced for arbitrary thermo-mechanical processes in (3.16) can be represented in terms of the functions ρ and $\hat{\psi}$ characteristic of the material as follows:

$$f = \rho[[\hat{\psi}(\underset{\sim}{F},\theta)]] - (1/2)\rho\left[\hat{\psi}_{\underset{\sim}{F}}(\overset{+}{\underset{\sim}{F}},\theta) + \hat{\psi}_{\underset{\sim}{F}}(\overset{-}{\underset{\sim}{F}},\theta)\right]\cdot[[\underset{\sim}{F}]] \ , \tag{5.7}$$

where $\overset{+}{\underset{\sim}{F}} = \overset{+}{\underset{\sim}{F}}(\cdot,t)$ and $\overset{-}{\underset{\sim}{F}} = \overset{-}{\underset{\sim}{F}}(\cdot,t)$ represent the limiting values of the deformation gradient tensor on the positive and negative sides of S(t), respectively.

5.2 Isothermal processes in thermoelastic materials. For an isothermal process, (4.1) and (3.11) show that $\Gamma_{con} = 0$, so that using (3.17), one may further reduce (5.4) to

$$\Gamma(t;D) = \Gamma_s(t;D) = (1/\theta_0)\int_{S(t)\cap D} f\, V_n\, dA \ . \tag{5.8}$$

Thus for isothermal processes in thermoelastic materials, entropy production arises solely from the motion of the surface of discontinuity S(t). The local consequence (3.19) of the second law at points on the singular surface of course continues to hold.

When considering isothermal processes in thermo-elastic materials, it is convenient to introduce the elastic potential W at the temperature θ_0 by

setting

$$W(\underset{\sim}{F}) = \rho\hat{\psi}(\underset{\sim}{F},\theta_0) \tag{5.9}$$

for every tensor $\underset{\sim}{F}$ with positive determinant. Then $(5.6)_1$ reads

$$\underset{\sim}{g}(\underset{\sim}{x},t) = W_{\underset{\sim}{F}}(\underset{\sim}{F}(\underset{\sim}{x},t)) \quad \text{for } \underset{\sim}{x}\in R\text{-}S(t),\ t\in[t_0,t_1]. \tag{5.10}$$

The remaining <u>mechanical</u> field equations $(2.7)_{1,2}$ are

$$\left.\begin{aligned} \text{Div } \underset{\sim}{g} + \rho\underset{\sim}{b} &= \rho\dot{\underset{\sim}{v}}\ , \\ \underset{\sim}{g}\underset{\sim}{F}^T &= \underset{\sim}{F}\underset{\sim}{g}^T\ , \end{aligned}\right\} \quad \text{on } R\text{-}S(t), \tag{5.11}$$

while the mechanical jump conditions are $(2.10)_2$ and $(2.8)_1$:

$$[[\underset{\sim}{v}]] = -\,[[\underset{\sim}{F}]]\underset{\sim}{V}\ , \quad [[\underset{\sim}{g}\underset{\sim}{n}]] + \rho[[\underset{\sim}{v}]]V_n = \underset{\sim}{0} \quad \text{on } S(t); \tag{5.12}$$

(5.10) - (5.12) comprise the usual field equations and jump conditions of elastodynamics. From (5.7), (5.9), the shock driving traction f may be written in the form

$$f = [[W(\underset{\sim}{F})]] \ - (1/2)\left[\ W_{\underset{\sim}{F}}(\overset{+}{\underset{\sim}{F}}) + W_{\underset{\sim}{F}}(\overset{-}{\underset{\sim}{F}})\ \right]\cdot[[\underset{\sim}{F}]]\ . \tag{5.13}$$

The inequality (3.19) must also hold:

$$f\, V_n \geq 0 \qquad \text{on} \quad S(t). \tag{5.14}$$

5.3 Equilibrium states and isothermal quasi-static processes in thermoelastic materials. Assume that the material is thermoelastic, and suppose that the displacement field $\underset{\sim}{u}$ in (2.1) is independent of time. Suppose further that the temperature θ is constant, so that (4.1) holds. Among the field equations (2.7), the purely mechanical ones are now

$$\left.\begin{aligned} &\text{Div}\, \underset{\sim}{\sigma} + \rho \underset{\sim}{b} = \underset{\sim}{0}\,, \\ &\underset{\sim}{\sigma}\underset{\sim}{F}^{T} = \underset{\sim}{F}\underset{\sim}{\sigma}^{T}, \end{aligned}\right\} \quad \text{on R-S.} \tag{5.15}$$

In addition, there is the mechanical constitutive requirement (5.10):

$$\underset{\sim}{\sigma} = W_{\underset{\sim}{F}}(\underset{\sim}{F}) \qquad \text{on R-S.} \tag{5.16}$$

The system (5.15), (5.16) with $\underset{\sim}{F} = \underset{\sim}{1} + \nabla\underset{\sim}{u}$ comprises the field equations of elastostatics. When this system fails to be elliptic, it may possess weak solutions for which $\underset{\sim}{F}$ jumps across a singular surface S; see Knowles and Sternberg (1978), Gurtin (1983), Rosakis (1988). Such solutions must satisfy the purely mechanical jump conditions $(2.10)_1$ and $(2.8)_1$ specialized to the equilibrium case:

$$[[\underset{\sim}{\sigma}\underset{\sim}{n}]] = \underset{\sim}{0}\,, \quad [[\underset{\sim}{F}]]\underset{\sim}{\ell} = \underset{\sim}{0} \qquad \text{on S.} \tag{5.17}$$

Because of the form of the jump condition $(5.17)_1$ in the equilibrium case, the formula (5.13) for the shock traction may be simplified to give

$$f = [[W(\underset{\sim}{F})]] - W_{\underset{\sim}{F}}(\underset{\sim}{\overset{+}{F}})\cdot[[\underset{\sim}{F}]] \ , \qquad \text{(static case)}. \tag{5.18}$$

This representation of the shock driving traction for equilibrium deformations of an elastic solid is equivalent to one derived by Yatomi and Nishimura (1983). An alternative formula for f in the present static case is readily shown to be

$$f = \underset{\sim}{n}\cdot[[\underset{\sim}{P}(\underset{\sim}{F})]]\underset{\sim}{n} \ , \qquad \text{(static case)}. \tag{5.19}$$

where $\underset{\sim}{n}$ is the unit normal to the surface of discontinuity S, and $\underset{\sim}{P}(\underset{\sim}{F})$ is the energy-momentum tensor introduced by Eshelby (1956,1970, 1975):

$$\underset{\sim}{P}(\underset{\sim}{F}) = W(\underset{\sim}{F})\underset{\sim}{1} - \underset{\sim}{F}^{T}W_{\underset{\sim}{F}}(\underset{\sim}{F}) \ ; \tag{5.20}$$

here $\underset{\sim}{1}$ is the identity tensor. In the form (5.19),(5.20) appropriate to the equilibrium case, the entity f first appeared as the "force on a defect" in the work of Eshelby (1956). This representation for f is also equivalent to that for the force on the interface between two phases derived by Eshelby (1970) and discussed by Rice (1975). The formulas (5.19), (5.20) may also be found in Knowles (1979).

An isothermal quasi-static process on a time interval $[t_0,t_1]$ is a one-parameter family of equilibrium fields $\underset{\sim}{u}(\cdot,t)$ and $\underset{\sim}{\sigma}(\cdot,t)$ with smoothness as specified in Section 2 and satisfying (5.15) - (5.17) at each $t\in[t_0,t_1]$. When at each instant t the fields involve a singular surface S(t) across which $\underset{\sim}{F}(\cdot,t)$ jumps, the inequality (5.14) is imposed as a requirement, f being given by (5.18) at each t.

6. Kinetic relations.

Using the one-dimensional, isothermal theory of bars of non-elliptic elastic material in tension (Abeyaratne and Knowles (1988a,b)), we have considered a bar lying along the x-axis in an equilibrium state in which there is a strain discontinuity located at an arbitary station $x = s$. The force F acting on the bar is assumed to be given, and the stress response of the material is taken to be one for which the stress at first rises with increasing strain, then declines, and finally rises again. For each F in a certain range, there is a one parameter family, parameter s, of such equilibrium states of physical interest. For each such state, the overall elongation e of the bar depends on both F and s, as do other macroscopic quantities such as the total strain energy E and the total potential energy $U = E - Fe$ in the bar: $e = e(F,s)$, $E = E(F,s)$, $U = U(F,s)$. It turns out that, in this equilibrium theory, $\partial U(F,s)/\partial F = -e$, and $\partial U(F,s)/\partial s = Af$, where A is the cross-sectional area of the bar, and f is the one-dimensional counterpart of the static driving traction given in (5.18). This suggests that the location s of the strain discontinuity plays the role of an "internal variable" whose "conjugate force" is proportional to the driving traction f. In the references cited above, quasi-static pro-

cesses are considered in which at each instant the bar is in one of the equilibrium states just described. Admissible quasi-static processes are those that statisfy the one-dimensional counterpart $f\dot{s} \geq 0$ of (5.14). Even with admissibility imposed as a requirement, specifying the force history $F(\tau)$, $0 \leq \tau \leq t$, in a quasi-static motion fails to determine the current value of the elongation $e(t)$, since the shock location (or internal variable) $s(t)$, must be specified as well. This suggests a constitutive deficiency. In internal variable theories designed to model microstructural effects on inelastic macrostructural behavior, a so-called "kinetic relation" giving the time rate of change of the internal, or microstructural, variable as a function of the associated "thermodynamic force" is commonly added as a part of the constitutive description; see, for example, Rice (1970, 1971, 1975). In their one-dimensional setting, Abeyaratne and Knowles (1988a,b) adopt this point of view and hence require that the shock location $s(t)$ and the associated driving traction $f(t)$ be related by a kinetic law of the form $\dot{s}(t) = V(f(t))$, where the kinetic response function V is determined by the material. Such a kinetic relation then couples the time-evolution of the location of the surface of strain discontinuity to the local strains on either side of the jump.

The isothermal equilibrium problem of the twisting of an infinite medium containing a circular hole considered by Abeyaratne and Knowles (1987a,b) involves cylindrical geometry in plane strain and is essentially one-dimensional because of axial symmetry. The surface of discontinuity is now circular, and its radius plays the role of the location s of the strain jump in the tensile bar problem. The formalism appropriate to internal

variables emerges in this problem as well, and an additional constitutive postulate in the form of a kinetic relation may be proposed here on similar grounds. It would now take the form

$$V_n(\underset{\sim}{x},t) = V(f(\underset{\sim}{x},t)), \quad \underset{\sim}{x}\in S(t), \tag{6.1}$$

where $\underset{\sim}{x}$ is the position vector of a typical point on the cylindrical surface of discontinuity $S(t)$, $f(\underset{\sim}{x},t)$ is given at each t by (5.18) specialized to plane strain for incompressible materials, and $V_n(\underset{\sim}{x},t)$ is the component of velocity of the singular surface normal to itself, i.e. in the radial direction. Again, $V(f)$ is a function determined by the material. To be consistent with the admissibility requirement (5.14), V must be such that

$$V(f)f \geq 0 \tag{6.2}$$

for all possible values of f.

The consequences of imposing a kinetic relation in the one-dimensional setting of the bar are discussed in detail in Abeyaratne and Knowles (1988a,b) and will not be repeated here. A similar discussion could be provided for the twist problem. While in both cases the motivation for the imposition of the kinetic relation as a requirement is taken from internal variable theories of inelastic behavior, the possibility of imposing it comes about because of the lack of uniqueness of weak solutions to the relevant equilibrium problem. Indeed, for a given force F in the bar problem, for example, solutions with a single "equilibrium shock" at $x = s$ fail

to be unique precisely to the extent that their totality comprises a one-parameter family (parameter s). This is exactly the extent of lack-of-uniqueness needed to make room for the kinetic law as an additional requirement. Thus the equilibrium problem seems to be under-determined to precisely the degree necessary to accommodate a kinetic relation applicable to isothermal quasi-static processes in the bar. Entirely similar remarks apply to the axially symmetric twist problem.

The general proposal of a constitutive relationship between the driving traction $f(\underset{\sim}{x},t)$ and the normal velocity $V_n(\underset{\sim}{x},t)$ of the material singular surface $S(t)$ during quasi-static processes in a thermoelastic material may be examined from the perspective of irreversible thermodynamics. In the representation for the entropy production rate Γ provided by (5.4), (3.11) and (3.17), one could view the ratios $(\text{Grad } \theta)/\theta^2$ and f/θ as thermodynamic "affinities" and the terms $\underset{\sim}{q}$ and V_n as the corresponding "fluxes"; see Chapter 14 of Callen (1985), Chapter 14 of Kestin (1968), and Lecture 7 of Truesdell (1969) for discussions of these notions. In the theory of irreversible processes, it is customary to postulate a constitutive relationship in which the present value of each flux is a function of the present value of the affinities and perhaps of their past histories as well. In our setting, one might - as a simplest case - postulate a relationship between the present values of each flux and its corresponding affinity, as in the theory of "purely resistive" thermodynamical systems (Callen (1985), Chapter 14). This would give

$$\underset{\sim}{q}(\underset{\sim}{x},t) = \underset{\sim}{Q}^*(\text{Grad } \theta(\underset{\sim}{x},t)/\theta^2(\underset{\sim}{x},t)), \quad \underset{\sim}{x}\in R\text{-}S(t), \tag{6.3}$$

and

$$V_n(\underset{\sim}{x},t) = V^*(f(\underset{\sim}{x},t)/\theta(\underset{\sim}{x},t)), \quad \underset{\sim}{x}\in S(t) , \tag{6.4}$$

where $\underset{\sim}{Q}^*$ and V^* are functions determined by the material; the constitutive statements (6.3) and (6.4) represent a heat conduction law and a kinetic relation, respectively. The consequence (5.14) of the Clausius-Duhem inequality would require that

$$V^*(f/\theta)f/\theta \geq 0 \tag{6.5}$$

for all possible values of f/θ. For isothermal processes of the type discussed in the preceding paragraphs of this section, (6.4) reduces to (6.1) with $V(f) = V^*(f/\theta_0)$, and (6.5) reduces to (6.2). We observe that, if the kinetic response function V^* (or, equivalently, V) is continuous, then (6.5) requires that

$$V^*(0) = V(0) = 0. \tag{6.6}$$

If the body is at a uniform temperature and in a state of mechanical equilibrium involving an equilibrium shock S at all points of which the Maxwell condition $f = 0$ holds, we say the body is in a Maxwell state. If V^* is smooth and departures from a Maxwell state are slight, so that f/θ is small, one might replace (6.4) by its linearization. (Linearized kinetic relations are often used in irreversible thermodynamics to describe pro-

cesses that are "close to thermodynamic equilibrium".) In view of (6.6), this would yield

$$V_n(\underset{\sim}{x},t) = \nu f(\underset{\sim}{x},t)/\theta(\underset{\sim}{x},t) \ , \qquad \underset{\sim}{x}\in S(t), \tag{6.7}$$

where $\nu = V'(0)$ is a material constant; by (6.2), ν is necessarily positive. A linear relation of the type (6.7) between the driving traction and the velocity of the singular surface was shown by Abeyaratne and Knowles (1988b) to lead to a conventional type of viscoelastic macroscopic response in the one-dimensional theory of quasi-static processes in tensile bars composed of a particular non-elliptic elastic material.

Note that the singular surface S(t) lies in the region occupied by the body in the reference configuration; thus the driving traction f might more precisely be called the nominal driving traction. The kinetic relation (6.4) is thus a "Lagrangian", or "material", assertion. It is readily shown that it conforms to the principle of material frame indifference.

For isothermal, quasi-static processes in a thermoelastic material, the state of the body at each instant is one of mechanical equilibrium, and an equilibrium shock cannot occur unless the corresponding elastic potential is non-elliptic. The equilibrium shock may be thought of as a phase boundary, and the quasi-static process may be viewed as one in which particles of the body are being transformed from one phase to another. The normal velocity V_n is clearly a measure of the rate at which this takes place, and the relation (6.4) may thus be regarded as controlling the kinetics of the

phase transformation; see, for example, Section 3.1 of Fine (1964) or Section 1.9 of Porter and Easterling (1981) for related discussions.

If the kinetic relation is to be constitutive, it must apply not only during quasi-static processes, but when inertial effects are included as well. For dynamical processes, it is to be expected that moving surfaces of strain discontinuity S(t) - counterparts of ordinary shock waves, for example - can occur which are not phase boundaries, and for which a supplementary relation such as (6.4) is thus expected to be inappropriate on physical grounds. From the mathematical point of view, it is the lack of uniqueness of solutions of equilibrium boundary value problems for non-elliptic elastic materials that allows the prescription of a supplementary kinetic relation for quasi-static processes. Whether there is a precisely analogous lack of uniqueness for the boundary-initial-value problems arising in the dynamics of non-elliptic elastic materials, and how in general to distinguish phase boundaries from ordinary shock waves, become questions of some importance. In the mathematical study of systems of conservation laws in one space dimension (see, for example, Dafermos (1983,1984), Lax (1973)), it is known that the solution to the initial-value problem subject to an "entropy inequality" such as $(2.8)_3$ (or, equivalently, (3.19)) is unique, provided that the curvature of the underlying stress-strain relation is always of one sign. In the absence of both convexity and concavity, the entropy inequality is not strong enough to secure uniqueness. The fact that the entropy inequality must be supplemented in order to assure uniqueness for the initial value problem for such materials reflects the need for an additional requirement of a constitutive nature,

as distinguished from a fundamental thermo-mechanical principle. A kinetic relation such as (6.4), in which f is now given by the representation (5.7) pertinent to dynamical processes and the material function V^* is subject to the requirement (6.5) imposed by the entropy inequality, might provide an appropriate supplementary constitutive requirement. Additional restrictions strong enough to guarantee uniqueness in the initial value problem in one space dimension for materials whose stress-strain relation is neither convex nor concave have been investigated from viewpoints different from the present one by a number of authors: see, for example, Hatori (1986), James (1980), Oleinik (1959), Shearer (1985) and Slemrod (1983).

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